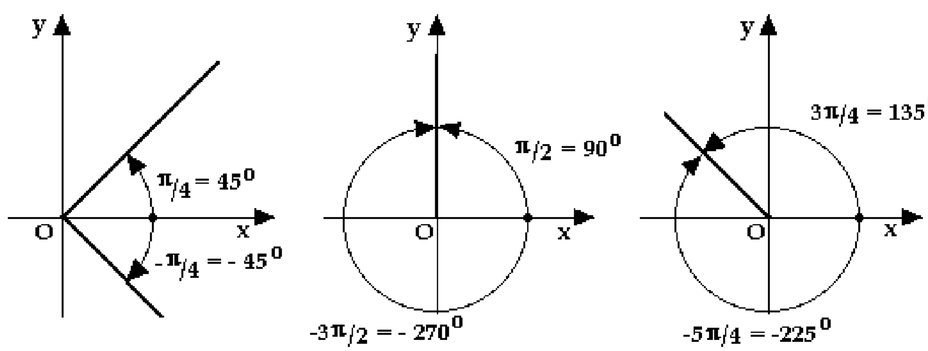
6. Trigonometry

This section contains review material on:

- Trigonometric ratios and trigonometric functions
- Trigonometric identities and trigonometric equations

Angles. Recall that a positive angle is measured counterclockwise from the direction of the positive x-axis. If it is measured clockwise, it is negative; see the figure below. The units commonly used are degrees (^{o}) and radians (rad). By convention, we use radians (unless stated otherwise). For example, $\sin 1$ denotes the value of the trigonometric function sine for 1 radian (using a calculator, we get $\sin 1 \approx 0.841471$).

A full revolution equals 360 degrees $= 2\pi$ radians. Thus, 1 degree equals $2\pi/360 = \pi/180$ radians (that is, to convert from degrees to radians, we multiply by $\pi/180$). Conversely, 1 radian equals $360/(2\pi) = 180/\pi$ degrees (and in order to convert radians into degrees, we multiply by $180/\pi$). For example, 90 degrees equals $90\frac{\pi}{180} = \frac{\pi}{2}$ radians. Similarly, $\frac{5\pi}{4}$ radians equals $\frac{5\pi}{4}\frac{180}{\pi} = 225$ degrees.



Exercise 1.

(a) Express 225 degrees in radians

(b) Express $\frac{7\pi}{6}$ radians in degrees.

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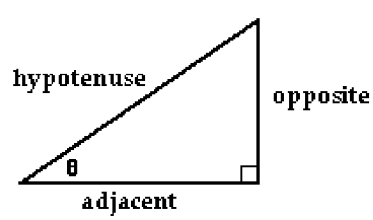
Trigonometric Ratios. For an acute angle, the trigonometric ratios are defined as ratios of lengths of sides in a right triangle.

Basic trigonometric ratios

sine: $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$

cosine: $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

tangent: $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\text{opposite}}{\text{adjacent}}$



The remaining three ratios are usually defined as the reciprocals of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

Trigonometric ratios

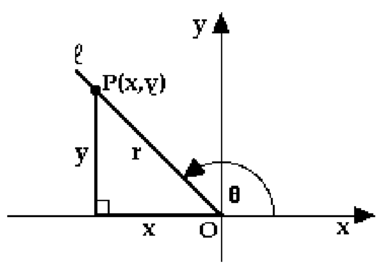
cosecant: $\csc\theta = \dfrac{1}{\sin\theta} = \dfrac{\text{hypotenuse}}{\text{opposite}}$

secant: $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{\text{hypotenuse}}{\text{adjacent}}$

cotanent: $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\text{adjacent}}{\text{opposite}}$

For general angles (such as obtuse or negative angles) the above definition does not apply, and we proceed as follows.

Let θ be an angle defined by the x-axis and a line ℓ, see the figure below.



Choose a point P anywhere on the line ℓ (as long as it is not the origin), and denote by (x, y) its coordinates. Let r be the distance between P and the origin (recall that $r = \sqrt{x^2 + y^2} > 0$). We define:

Trigonometric ratios for general angles

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{r}{y} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{r}{x} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}$$

Note that $\sin\theta$ and $\cos\theta$ are always defined. However, that is not true for the remaining four ratios. The ratios $\tan\theta$ and $\sec\theta$ are not defined when $x = 0$, and $\cot\theta$ and $\csc\theta$ are not defined when $y = 0$.

For an acute angle, the two definitions agree.

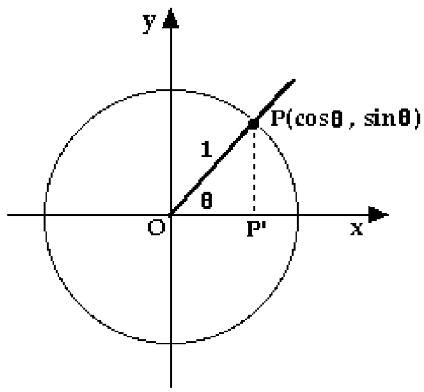
Note that

$$\sin^2\theta + \cos^2\theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{x^2 + y^2}{r^2} = 1,$$

since $x^2 + y^2 = r^2$. Thus, we have obtained the basic trigonometric identity:

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

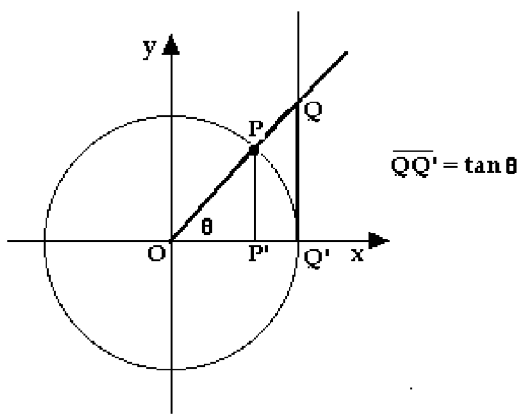
It is also possible to use a unit circle to define trigonometric ratios. Let P be the point of intersection of a circle of radius 1 and the line whose angle (positive or negative) with respect to the x-axis is θ; see the figure below.



By definition, the coordinates of P are $(\cos\theta, \sin\theta)$. In other words, $\overline{OP'} = \cos\theta$ and $\overline{PP'} = \sin\theta$. Now, draw the vertical line that intersects the x-axis at $(1, 0)$ and label its intersection with the line OP by Q, see the figure below. The triangles OPP' and OQQ' are similar. Thus,

$$\frac{\overline{QQ'}}{\overline{OQ'}} = \frac{\overline{PP'}}{\overline{OP'}} \qquad \text{implies} \qquad \frac{\overline{QQ'}}{1} = \frac{\sin\theta}{\cos\theta}.$$

Thus, $\overline{QQ'} = \tan\theta$.

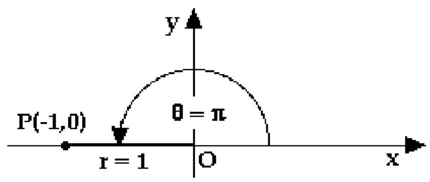


Values of Trigonometric Ratios for Special Angles.

(1) $\theta = 0$ (radians). Looking at the unit circle we see that, when $\theta = 0$, $\overline{OP'} = 1$ and $\overline{PP'} = 0$; in other words, $\cos 0 = 1$ and $\sin 0 = 0$. Consequently, $\tan 0 = \sin 0/\cos 0 = 0$, and $\sec 0 = 1/\cos 0 = 1$. The ratios $\cot 0$ and $\csc 0$ are not defined.

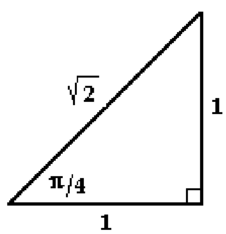
(2) $\theta = \pi/2$ (radians). In this case (use the unit circle definition again), $\overline{OP'} = 0$ and $\overline{PP'} = 1$ (in other words, the coordinates of P are $(0,1)$). Thus, $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$. It follows that $\tan\frac{\pi}{2}$ and $\sec\frac{\pi}{2}$ are not defined. Finally, $\cot\frac{\pi}{2} = \cos\frac{\pi}{2}/\sin\frac{\pi}{2} = 0/1 = 0$ and $\csc\frac{\pi}{2} = 1/\sin\frac{\pi}{2} = 1$.

(3) $\theta = \pi$ (radians). For a change, we use the definition for general angles: pick a point $P(x = -1, y = 0)$; then $r = \sqrt{(-1)^2 + 0^2} = 1$.



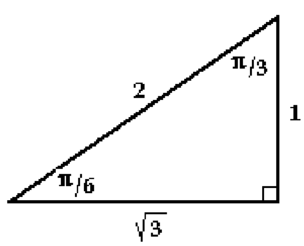
It follows that $\cos\pi = x/r = -1/1 = -1$ and $\sin\pi = y/r = 0/1 = 0$. Consequently, $\tan\pi = 0$ and $\sec\pi = -1$. The ratios $\cot\pi$ and $\csc\pi$ are not defined.

(4) $\theta = \pi/4$ (radians). The values of trigonometric ratios can be read from the triangle below.



From the definition for acute angles, we get $\sin\frac{\pi}{4} = $ opposite / hypotenuse $= 1/\sqrt{2}$, $\cos\frac{\pi}{4} = $ adjacent / hypotenuse $= 1/\sqrt{2}$ and $\tan\frac{\pi}{4} = $ opposite / adjacent $= 1$.

(5) $\theta = \pi/6$ and $\theta = \pi/3$ (radians). The values of trigonometric ratios can be read from the triangle below.



From the definition for acute angles, we get

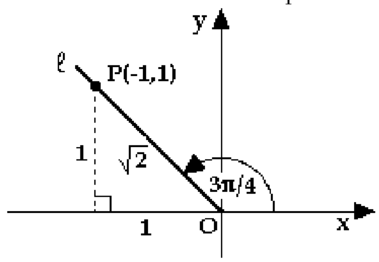
$$\sin\frac{\pi}{6} = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2}, \quad \cos\frac{\pi}{6} = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2} \quad \text{and} \quad \tan\frac{\pi}{6} = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{\sqrt{3}}.$$

Similarly, $\sin\frac{\pi}{3} = \sqrt{3}/2$, $\cos\frac{\pi}{3} = 1/2$ and $\tan\frac{\pi}{3} = \sqrt{3}$.

Example 1. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ for
(a) $\theta = 3\pi/4$ (b) $\theta = 2\pi/3$.

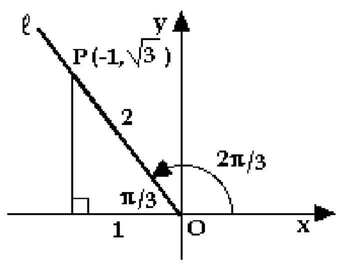
Solution

(a) We use the definition for general angles. The line that makes the angle of $\theta = \frac{3\pi}{4}$ radians with respect to the x-axis is a line with slope -1. Thus, we can choose the point $(x = -1, y = 1)$ as P.



In that case, $r = \sqrt{1+1} = \sqrt{2}$, and it follows that $\sin\frac{3\pi}{4} = y/r = 1/\sqrt{2}$, $\cos\frac{3\pi}{4} = x/r = -1/\sqrt{2}$ and $\tan\frac{3\pi}{4} = y/x = -1$.

(b) We use the definition for general angles. Placing the triangle that we used to compute the ratios for $\pi/6$ and $\pi/3$ (see (5) above), as shown in the figure, we see that we can use the point $(x = -1, y = \sqrt{3})$ as P.



It follows that $r = \sqrt{x^2+y^2} = \sqrt{1+3} = 2$, and thus $\sin\frac{2\pi}{3} = y/r = \sqrt{3}/2$, $\cos\frac{2\pi}{3} = x/r = -1/2$ and $\tan\frac{2\pi}{3} = \sqrt{3}$. ∎

Exercise 2. Find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ for
(a) $\theta = 5\pi/6$ (b) $\theta = -\pi/6$.

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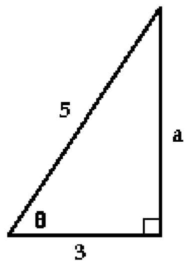
Example 2. If $0 < \theta < \pi/2$, and $\cos\theta = 3/5$, find the values of $\sin\theta$, $\tan\theta$ and $\sec\theta$.

Solution

By definition, $\sec\theta = 1/\cos\theta = 5/3$.

Using the fact that $\cos\theta$ is the ratio of the adjacent side to the hypotenuse in an acute triangle (and that is given, since $0 < \theta < \pi/2$!), we label the triangle as follows:



By Pythagorean theorem, the opposite side is equal to $a = \sqrt{5^2 - 3^2} = 4$. Thus, $\sin\theta = a/5 = 4/5$ and $\tan\theta = a/3 = 4/3$. ∎

Exercise 3. If $0 < \theta < \pi/2$, and $\csc\theta = 3$, find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

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Exercise 4. Find the values of $\sin\theta$, $\cos\theta$, $\tan\theta$ and $\sec\theta$ for $\theta = -3\pi/2$.

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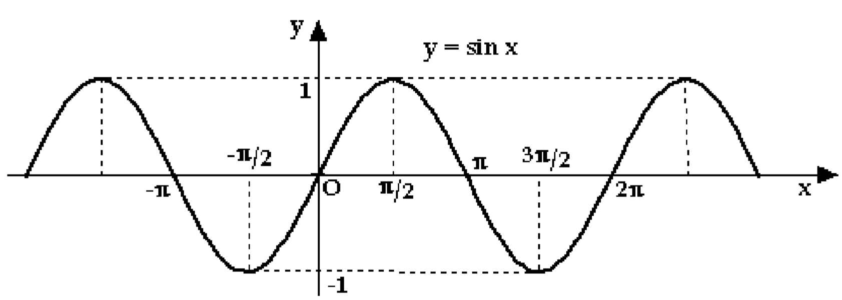
Trigonometric Functions. Let x denote an angle (in radians). Using the general method of defining trigonometric ratios, we can compute the values of the functions $y = \sin x$ and $y = \cos x$ for all real numbers x (keep in mind that x denotes an angle in radians).

Since the angles x and $x + 2\pi$ are the same (think of an angle and what it looks like one full revolution later), it follows that

Periodicity of $\sin x$ and $\cos x$
$\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x$

These formulas state that the values of sin and cos repeat after 2π radians. In other words, $\sin x$ and $\cos x$ are periodic with period equal to 2π.

By plotting points we obtain the graphs of the two functions. Given below is the graph of $y = \sin x$.

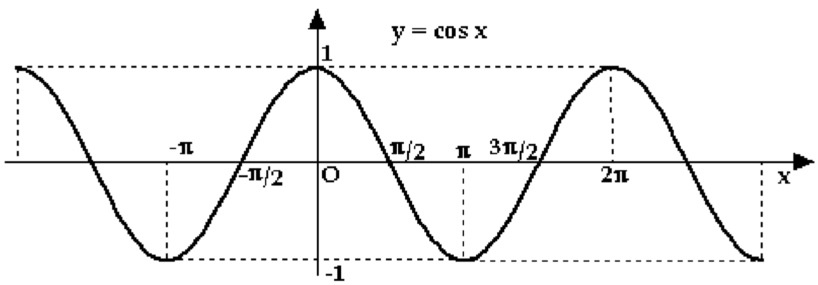


The part of the graph of $\sin x$ over the interval $[0, 2\pi]$ is called the main period. That part is repeated in both directions to produce the whole graph.

Note that $\sin x = 0$ when $x = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$. In words, $\sin x = 0$ when x is an integer multiple of π, i.e., when $x = k\pi$ (k is an integer). We have to remember this fact.

$$\boxed{\sin x = 0 \quad \text{if and only if} \quad x = k\pi \ (k = \text{integer})}$$

Given below is the graph of $y = \cos x$.



Note that $\cos x = 0$ when $x = \ldots, -3\pi/2, -\pi/2, \pi/2, 3\pi/2, \ldots$. In words, $\cos x = 0$ at $\pi/2$ and all points that are a multiple of π away from it. Thus,

$$\boxed{\cos x = 0 \quad \text{if and only if} \quad x = \frac{\pi}{2} + k\pi \ (k = \text{integer})}$$

The part of the graph of $\cos x$ over the interval $[0, 2\pi]$ is called the main period. That part is repeated in both directions to produce the whole graph.

Note that

$$\boxed{-1 \leq \sin x \leq 1 \quad \text{and} \quad -1 \leq \cos x \leq 1}$$

Recall the basic trigonometric identity

$$\boxed{\sin^2 x + \cos^2 x = 1}$$

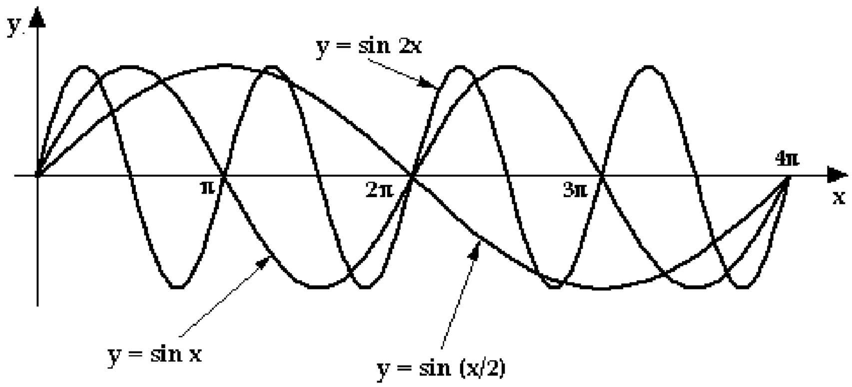
Example 3. Sketch the graphs of $y = \sin x$, $y = \sin 2x$ and $y = \sin(x/2)$ in the same coordinate system.

Solution

(a) Recall that the main period of $\sin x$ is defined to be the interval from $x = 0$ to $x = 2\pi$. Replacing x by $2x$, we get that the main period of $\sin 2x$ is the interval from $2x = 0$ (i.e., $x = 0$) to $2x = 2\pi$ (i.e., $x = \pi$). In other words, the period of $\sin 2x$ is π.

Rephrasing the above argument, we can show that the period of $\sin(ax)$ is $2\pi/a$.

Thus, the graph of $\sin 2x$ is obtained from the graph of $\sin x$ by compressing it along the x-axis by the factor of 2. The period of $\sin(x/2)$ is $2\pi/(1/2) = 4\pi$. Thus, its graph is obtained by stretching $\sin x$ along the x-axis by a factor if 2. See the figure below.



∎

Exercise 5. What is the period of $\cos(ax)$?

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Exercise 6. Sketch the graphs of $y = \cos x$, $y = \cos 3x$ and $y = \cos 0.5x$ in the same coordinate system.

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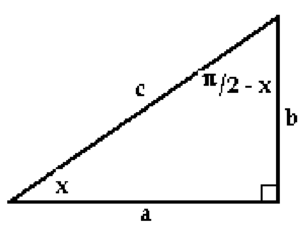
Consider the following triangle.



Using the definition of sin and cos, we get $\cos x = a/c$, $\sin x = b/c$, $\cos\left(\frac{\pi}{2} - x\right) = b/c$ $\sin\left(\frac{\pi}{2} - x\right) = a/c$. We have thus obtained the following formulas.

$$\boxed{\sin\left(\frac{\pi}{2} - x\right) = \cos x \quad \text{and} \quad \cos\left(\frac{\pi}{2} - x\right) = \sin x}$$

Next, we list useful formulas involving $\sin x$ and $\cos x$.

Relation between x and $-x$

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x$$

Addition and subtraction formulas

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

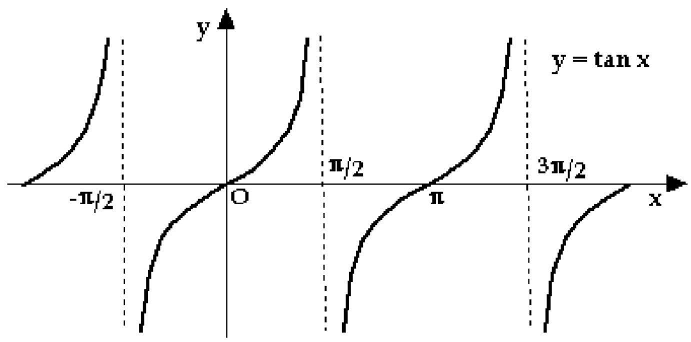
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

Double angle formulas

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

The function $y = \tan x = \frac{\sin x}{\cos x}$ is not defined when $\cos x = 0$; i.e., it is not defined when $x = \frac{\pi}{2} + k\pi$. The graph of $y = \tan x$ is given below.



$y = \tan x$ is periodic with period π. The part of the graph over the interval $(-\pi/2, \pi/2)$ is its main period. $y = \tan x = 0$ whenever $\sin x = 0$.

Exercise 7. What is the period of $\tan(ax)$?

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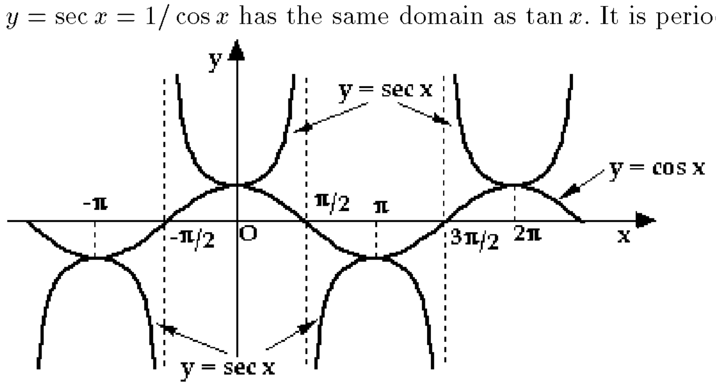
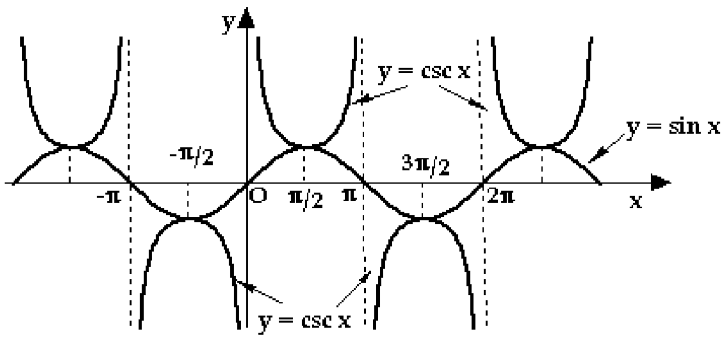
The function $y = \sec x = 1/\cos x$ has the same domain as $\tan x$. It is periodic with period 2π.



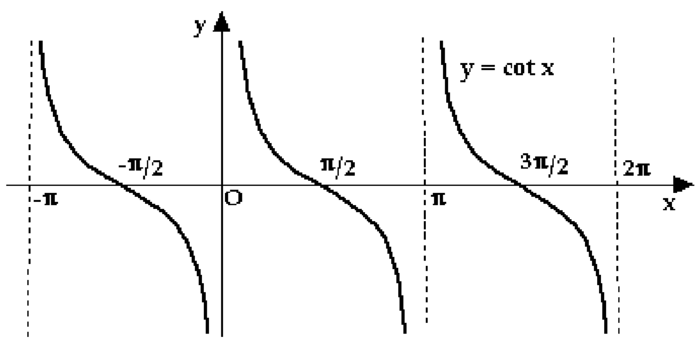
There is a useful relationship between the tangent and the secant, given by

$$\boxed{\tan^2 x + 1 = \sec^2 x}$$

The graph of $\csc x = 1/\sin x$ is given below.



The graph of $\cot x = 1/\tan x = \cos x/\sin x$ is given below. The domain of both $\csc x$ and $\cot x$ consists of all x such that $x \neq k\pi$ (k=integer).



Example 4. Prove the following formulas.

(a) $\sin(\pi - x) = \sin x$ (b) $(\sin x + \cos x)^2 = 1 + \sin 2x$

(c) $\dfrac{1}{1-\sin x} + \dfrac{1}{1+\sin x} = \dfrac{2}{\cos^2 x}$

Solution

(a) Using the subtraction formula for sin, we get

$$\sin(\pi - x) = \sin\pi\cos x - \cos\pi\sin x = 0\cdot\cos x - (-1)\sin x = \sin x.$$

(b) Squaring the left side,

$$(\sin x + \cos x)^2 = \sin^2 x + 2\sin x\cos x + \cos^2 x,$$

using $\sin^2 x + \cos^2 x = 1$ and the double angle formula for sin, we get

$$= 1 + 2\sin x\cos x = 1 + \sin 2x.$$

(c) Computing the common denominator on the left side, we get

$$\frac{1}{1-\sin x}+\frac{1}{1+\sin x}=\frac{1+\sin x}{1-\sin^2 x}+\frac{1-\sin x}{1-\sin^2 x}=\frac{2}{1-\sin^2 x}=\frac{2}{\cos^2 x}.$$

We used the identity $\sin^2 x+\cos^2 x=1$. ∎

Exercise 8. Prove the following formulas.
(a) $\sin(\pi/2+x)=\cos x$ (b) $\tan^2 x+1=\sec^2 x$
(c) $\sin^2 x-\tan^2 x+\sin^2 x\tan^2 x=0$.

♣

Example 5. Using addition formulas, prove the following identities.
(a) $\sin 2x=2\sin x\cos x$ (b) $\cos 2x=1-2\sin^2 x$.

Solution

(a) $\sin 2x=\sin(x+x)=\sin x\cos x+\cos x\sin x=2\sin x\cos x$.

(b) As in (a),

$$\cos 2x=\cos(x+x)=\cos x\cos x-\sin x\sin x=\cos^2 x-\sin^2 x$$

now use the identity $\sin^2 x+\cos^2 x=1$ to eliminate $\cos^2 x$

$$=(1-\sin^2 x)-\sin^2 x=1-2\sin^2 x.$$ ∎

Example 6. Show that $\cos 3x=4\cos^3 x-3\cos x$.

Solution

Write $3x=2x+x$, and start with the addition formula for cos :

$$\begin{aligned}\cos 3x&=\cos(2x+x)\\&=\cos 2x\cos x-\sin 2x\sin x\end{aligned}$$

use the double angle formulas

$$\begin{aligned}&=(2\cos^2 x-1)\cos x-(2\sin x\cos x)\sin x\\&=2\cos^3 x-\cos x-2\cos x\sin^2 x\end{aligned}$$

replace $\sin^2 x$ using the basic trigonometric identity $\sin^2 x=1-\cos^2 x$

$$\begin{aligned}&=2\cos^3 x-\cos x-2\cos x(1-\cos^2 x)\\&=2\cos^3 x-\cos x-2\cos x+2\cos^3 x\\&=4\cos^3 x-3\cos x.\end{aligned}$$ ∎

Exercise 9. Using the idea of the previous example, derive a formula that expresses $\sin 3x$ in terms of $\sin x$.

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Exercise 10. Show that $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y$.

♣

In what follows, the symbol k denotes an integer.

Trigonometric Equations. To find a solution to a trigonometric equation, we find all solutions in the main period first, and then add the multiple of the period.

Example 7. Solve the following equations.
(a) $\sin x = 1$ (b) $\tan x = 1$.

Solution

(a) Looking at the graph of $\sin x$, we see that $x = \frac{\pi}{2}$ is the only solution of $\sin x = 1$ in the main period of $\sin x$. Thus, all solutions are given by $x = \frac{\pi}{2} + 2k\pi$.

(b) There is only one solution to $\tan x = 1$ in the main period of tangent (which is the interval from $-\pi/2$ to $\pi/2$): $x = \frac{\pi}{4}$. Since the period of the tangent is π, all solutions are given by $x = \frac{\pi}{4} + k\pi$. ∎

Exercise 11. Solve the following equations.
(a) $\cos x = -1$ (b) $\tan x = -1$.

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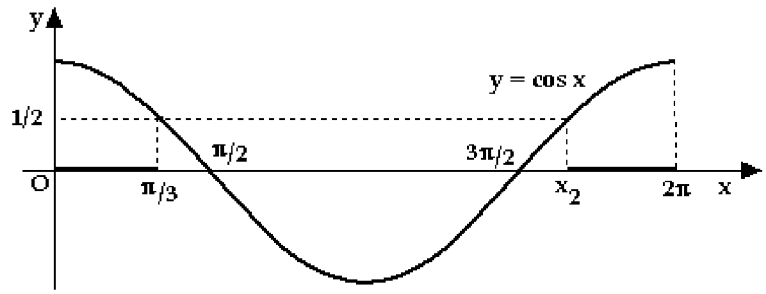
Example 8. Solve the following equations.
(a) $\cos x = 1/2$ (b) $\sin x = -1/2$.

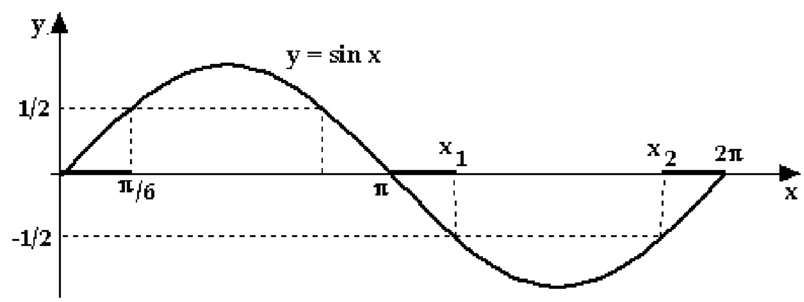
Solution

(a) From the graph below we see that there are two solutions of the given equation in the main period.



One of them is $x_1 = \frac{\pi}{3}$. Due to symmetry, the other solution is $\pi/3$ units to the left of 2π; thus, $x_2 = 2\pi - \pi/3 = 5\pi/3$. It follows that all solutions are given by $x = \frac{\pi}{3} + 2k\pi$ and $x = \frac{5\pi}{3} + 2k\pi$ (k is an integer).

(b) From the graph we see that there are two solutions of $\sin x = -1/2$ in the main period.



We know that $\sin \frac{\pi}{6} = \frac{1}{2}$. By symmetry, x_1 is $\pi/6$ units to the right of π, so $x_1 = \pi + \pi/6 = 7\pi/6$. Again, by symmetry, x_2 is $\pi/6$ units to the left of 2π; thus, $x_2 = 2\pi - \pi/6 = 11\pi/6$. Thus, the solutions are $x = \frac{7\pi}{6} + 2k\pi$ and $x = \frac{11\pi}{6} + 2k\pi$. ∎

Exercise 12. Solve the following equations.

(a) $\cos x = -\sqrt{3}/2$ (b) $\tan x = \sqrt{3}$ (c) $\sin x = \sqrt{2}/2$

♣

Example 9. Solve the equation $\sin 2x = \cos x$.

Solution

Using the double angle formula, we get

$$\sin 2x = \cos x$$
$$2 \sin x \cos x = \cos x$$
$$\cos x(2 \sin x - 1) = 0.$$

Thus, $\cos x = 0$ or $2\sin x - 1 = 0$. If $\cos x = 0$, then $x = \frac{\pi}{2} + k\pi$ (this equation has been solved earlier). If $\sin x = 1/2$, then $x = \frac{\pi}{6} + 2k\pi$ and $x = \frac{5\pi}{6} + 2k\pi$ (look at the graph of Example 8(b)). Thus, the solution is $x = \frac{\pi}{2} + k\pi$, $x = \frac{\pi}{6} + 2k\pi$ and $x = \frac{5\pi}{6} + 2k\pi$. ∎

Exercise 13. Solve the equation $\sin 2x = \sin x$.

♣

Example 10. Solve the equation $2 + \cos 2x = 3\cos x$.

Solution

Using the double angle formula for $\cos x$, we rewrite $2 + \cos 2x = 3\cos x$ as $2 + 2\cos^2 x - 1 = 3\cos x$. Thus $2\cos^2 x - 3\cos x + 1 = 0$, and $(2\cos x - 1)(\cos x - 1) = 0$.

It follows that $2\cos x - 1 = 0$, and $\cos x = 1/2$ (in which case $x = \frac{\pi}{3} + 2k\pi$ and $x = \frac{5\pi}{3} + 2k\pi$, see Example 8(a)) and $\cos x - 1 = 0$, and $\cos x = 1$ (in which case $x = 2k\pi$). Thus, the solution is $x = 2k\pi$, $x = \frac{\pi}{3} + 2k\pi$ and $x = \frac{5\pi}{3} + 2k\pi$. ∎

The latter two examples show that, if an equation contains different arguments of trig functions (such as sin and/or cos of x and $2x$), it is a good idea to reduce the expressions to a single argument (which is usually x).

Example 11. Solve the equation $4\sin 2x \cos 2x = 1$.

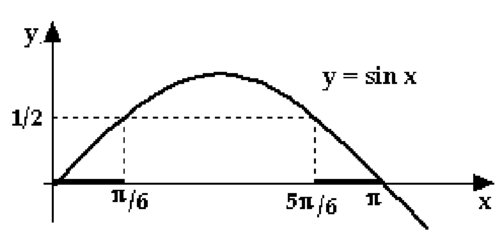
Solution

Using the double angle formula for $\sin x$, we get

$$4\sin 2x \cos 2x = 2(2\sin 2x \cos 2x) = 2\sin 4x = 1,$$

and thus $\sin 4x = 1/2$.

Now, $\sin A = 1/2$ implies $A = \pi/6$ or $A = 5\pi/6$, see the figure below.



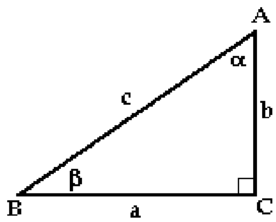
Thus (replacing A by $4x$), $4x = \frac{\pi}{6} + 2k\pi$ and $4x = \frac{5\pi}{6} + 2k\pi$, and the solutions are $x = \frac{\pi}{24} + \frac{k\pi}{2}$ and $x = \frac{5\pi}{24} + \frac{k\pi}{2}$. ∎

Exercise 14. Solve the equation $2\cos 2x - 1 = 0$.

♣

Additional exercises.

Exercise 15. Let ABC be a right triangle, where $\angle C = 90^0$; see the figure below.



(a) Given that $a = 21/5$ and $b = 4$, find $\sin\alpha$, $\cos\alpha$, $\sin\beta$, and $\cos\beta$.

(b) Given that $\cos\beta = 12/13$ and $c = 13$, find a, b, $\sin\beta$, and $\tan\beta$.

(c) Given that $c = 1$ and $a = 0.6$, find all six trigonometric ratios for angle β.

Exercise 16. What quadrant do the following angles belong to? (Recall the convention that states that if no units for angles are explicitly stated, then the units are radians.)

(a) $36\pi/7$ (b) 999^0 (c) 989^0 (d) $44\pi/5$.

Exercise 17. Without a calculator, determine the sign of the following expressions.

(a) $\tan(13\pi/3)$ (b) $\sin(500^0)$ (c) $\cos(37\pi)$ (d) $\sin(\pi/12) + \cos(\pi/7)$.

Exercise 18. Without a calculator, determine which of the following is larger.

(a) $\sin 1^0$ or $\sin 1$ (b) $\cos 2^0$ or $\cos 2$ (c) $\tan 1^0$ or $\tan 1$.

Exercise 19. Simplify the following expressions.

(a) $\sec^2 x - \sin^2 x - \cos^2 x$

(b) $\dfrac{\cos x}{1 + \sin x} + \tan x$

(c) $\dfrac{\sin x}{1 + \cos x} + \dfrac{\sin x}{1 - \cos x}$

(d) $(\sin x + \cos x)^2 + (\sin x - \cos x)^2$.

Exercise 20. Solve the following equations.

(a) $\tan x = -\sqrt{3}/3$

(b) $\cot x = -1$

(c) $\cos x = \sqrt{3}/2$.

Exercise 21. Solve the following equations.

(a) $\sin x = \sqrt{2}/2$

(b) $\tan x + \cot x = 0.5$

(c) $\cos^2 x - \cos x - 2 = 0$.

Exercise 22. Sketch the graphs of the following functions.

(a) $\cos(x + \pi/4)$ (b) $\sin(x - \pi)$ (c) $\tan(x + 1)$.

Exercise 23. Prove the following identities.

(a) $\dfrac{1 - \sin x}{\cos x} = \dfrac{\cos x}{1 + \sin x}$

(b) $\dfrac{1}{1 + \tan^2 x} + \dfrac{1}{1 + \cot^2 x} = 1$.